\MRI Primer, Exercise #7
Solution

Question 1

The signal at the echo time, as derived in class & in the notes, is given by

$$M_{xy}(\mathbf{r}) = \left[1 - e^{-TR/T_1(\mathbf{r})}\right] e^{-TE/T_2(\mathbf{r})} M_0(\mathbf{r})$$

The difference in this intensity between two types of tissue, differing only in their T_1, is given by

$$\begin{aligned} M_{xy}^{(2)}(\mathbf{r}) - M_{xy}^{(1)}(\mathbf{r}) &= \left[\left(1 - e^{-TR/T_1^{(2)}(\mathbf{r})}\right) - \left(1 - e^{-TR/T_1^{(1)}(\mathbf{r})}\right)\right] e^{-TE/T_2(\mathbf{r})} M_0(\mathbf{r}) \\ &= \left(e^{-TR/T_1^{(1)}(\mathbf{r})} - e^{-TR/T_1^{(2)}(\mathbf{r})}\right) e^{-TE/T_2(\mathbf{r})} M_0(\mathbf{r}) \end{aligned}$$

To maximize this, we need to find

$$\frac{d\left(M_{xy}^{(2)}(\mathbf{r}) - M_{xy}^{(1)}(\mathbf{r})\right)}{d(TR)} = 0$$

We've actually solved this equation in the lecture notes for the T_2* case in GRE experiments. The solution is:

$$TR_{\text{maximal contrast}} = \frac{T_1^{(1)} T_1^{(2)}}{\left(T_1^{(1)} - T_1^{(2)}\right)} \ln\left(\frac{T_1^{(1)}}{T_1^{(2)}}\right) \approx \frac{\left|T_1^{(2)} + T_1^{(2)}\right|}{2}$$

Question 2

The signal at echo time is

$$M_{xy}(\mathbf{r}) = \frac{\left[1 - e^{-TR/T_1(\mathbf{r})}\right] \sin(\alpha)}{1 - \cos(\alpha) e^{-TR/T_1(\mathbf{r})}} e^{-TE/T_2^*(\mathbf{r})} M_0(\mathbf{r})$$

To maximize it with respect to α, we need to find α for which

$$\frac{dM_{xy}}{d\alpha} = 0$$

Differentiating (this is a bit cumbersome), we obtain

$$\frac{(1-E_1)\cos(\alpha)}{1-E_1\cos(\alpha)} - \frac{(1-E_1)E_1\sin^2(\alpha)}{(1-E_1\cos(\alpha))^2} = 0, \qquad E_1 \equiv e^{-TR/T_1}$$

where I've denoted $E_1 = e^{-TR/T1}$. Simplifying, we obtain

$$\cos(\alpha) = \frac{E_1\sin^2(\alpha)}{1-E_1\cos(\alpha)}$$

Simplifying some more, and using $\cos^2+\sin^2=1$, we get the desired result

$$\cos(\alpha) = E_1 \,.$$

So, why isn't the Ernst angle used in MRI? Because, once a sufficient signal magnitude has been attained and you can see what you're after, increasing it further serves no purpose. Your goal in MRI is often **maximal contrast**, not maximal signal!

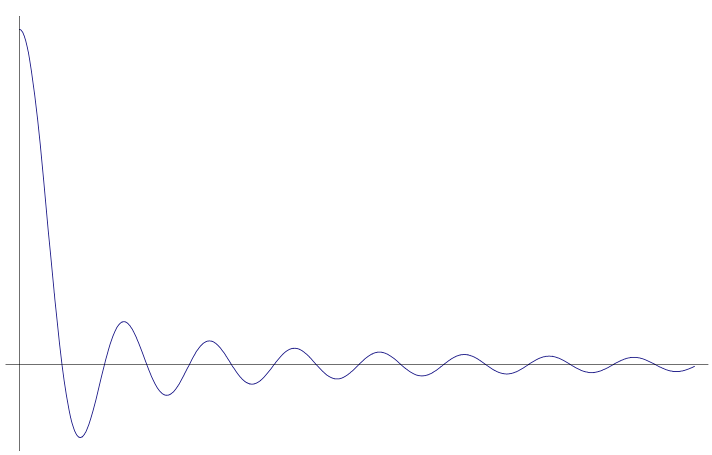
Question 3.
The magnetization is

$$M_{xy}(z,t) = M_0 e^{-i\gamma Gzt} \qquad \text{for } -L/2 \le z \le L/2$$

The signal is then

$$s(t) \propto \int_{-L/2}^{L/2} M_{xy}(z,t)\,dz = M_0 \int_{-L/2}^{L/2} e^{-i\gamma Gzt}\,dz = M_0 \frac{e^{i\gamma GLt/2} - e^{-i\gamma GLt/2}}{i\gamma Gt} = \frac{LM_0}{2}\text{sinc}\left[\frac{\gamma GLt}{2}\right]$$

where I've used $\sin(x)=(e^{ix}-e^{-ix})/2i$. So, the signal decays as follows:



The first 0 of a sinc function is encountered when

$$\text{sinc}(x) = 0 \qquad \text{when} \qquad \sin(x) = 0 \qquad \text{when} \qquad x = \pi/2$$

In our case, $x=\gamma GLt/2$, so the signal becomes 0 (for the first time) when

$$\frac{\gamma GLt}{2} = \pi/2 \qquad \Rightarrow \qquad t = \frac{\pi}{\gamma GL}$$

For a 30cm sample, with G = 40 mT/m, one has

$$t \approx \frac{1}{2\times 42\times 10^{6}\times 40\times 10^{-3}\times 0.3} \approx 1 \text{ microsecond} .$$