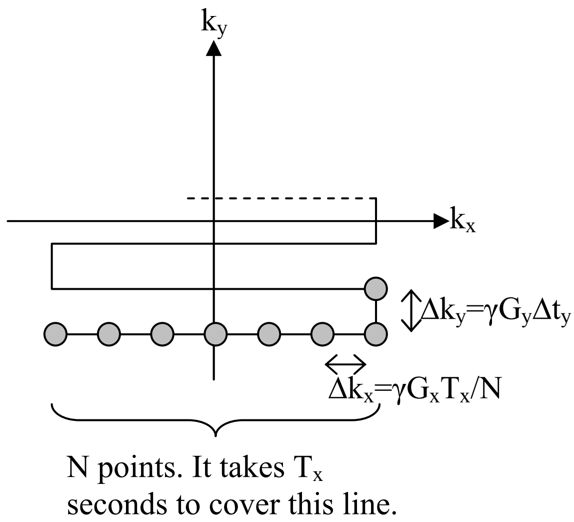
## MRI Primer
## Exercise #5: Solution

**Question 1.**

Thinking in terms of a trajectory in k-space (see figure on right), we must have $\Delta k_x=\Delta k_y=1/FOV_x=1/FOV_y=1/L$. Now,




$$\Delta k_x = \gamma G_x \times (\text{dwell time}) = \gamma G_x \times \frac{T_x}{N}$$

$$\Delta k_y = \gamma G_y \Delta t_y$$

where the dwell time – the time between each point sampled – is given by $T_x$ (the time it takes to sample a line) divided by N (the number of points in a line). Since, by choice, $\Delta k_x=\Delta k_y$, one obtains

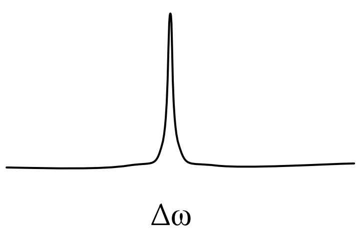
$$\Delta k_x = \gamma G_x T_x / N = \gamma G_y \Delta t_y = \Delta k_y$$

and since, by assumption, $T_x/N=\Delta t_y$, the desired result ($G_x=G_y$) is obtained.

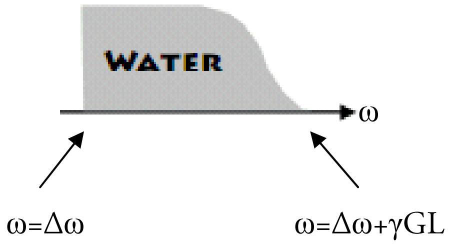
**Question 2.**

Think of what it means to acquire a 1D image of an object containing water, having some offset, $\Delta\omega$. Let's take the water profile in the question for concreteness:


In the absence of a gradient, all water spins have the same frequency, $\Delta\omega$. Acquiring (without a gradient) and Fourier-transforming would yield a peak at $\Delta\omega$:

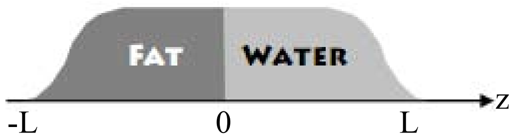


When you switch on a gradient, spins get a frequency assigned to them as a function of position: $\omega=\gamma Gz+\Delta\omega$. Acquiring in the presence of this gradient means you will get a peak from each position in the sample, which will give you the 1D profile of your object; the peak at $\omega$ will come from the spins at $z=\frac{\omega-\Delta\omega}{\gamma G}$:

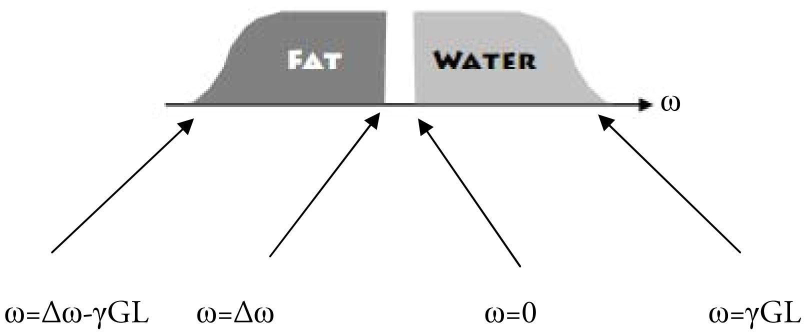



How would you translate this frequency axis to a position axis? By inverting the relation $\omega=\gamma Gz+\Delta\omega$ to yield $z(\omega)=\frac{\omega-\Delta\omega}{\gamma G}$. If you know $\Delta\omega$ (you do) and G, you can recover z.

For a sample with both fat and water, with $\Delta\omega=0$ for the water and $\Delta\omega\neq0$ for the fat (by assumption), one would get the following result (in frequency space) after Fourier transforming the signal:



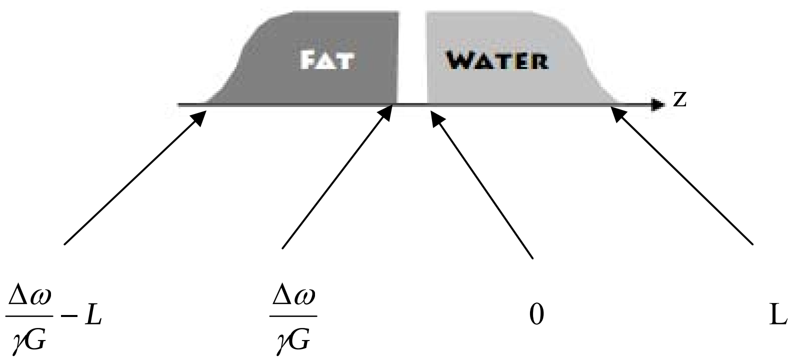

The object




You can see there's a $\Delta\omega$ gap in frequency space between the two profiles. When converting frequency to position, however, you have a problem. You don't know a-priori which peak comes from fat and which from water (here I've drawn the object so we do know a-priori which peak comes from fat and which peak comes from water, but when imaging unknown objects it's impossible to say). Therefore, we don't know $\Delta\omega$. We must assume $\Delta\omega=0$ (out of ignorance), so

$$\omega=\gamma Gz \qquad \Leftrightarrow \qquad z=\frac{\omega}{\gamma G}$$

Using this to "translate" between frequency & position, we obtain




So the shift, when expressed in terms of position, becomes

$$\Delta z = \frac{\Delta\omega}{\gamma G}$$

which is the magnitude of the shift between the fat and the water. The obvious way to **reduce** this artifact is to increase the gradient, G (making $\Delta z$ smaller).

There is also a formal, mathematical solution to the problem. Let $M_0(z)$ be the distribution of water or fat in the imaged object, with some chemical shift $\Delta\omega$. After exciting the spins, we have

$$M_{xy}(z) = M_0(z)$$

We apply a gradient -G for a time T/2, during which $\omega(z) = \Delta\omega - \gamma G z$, and after which

$$M_{xy}(z) = M_0(z) e^{-i\omega(z)T/2} = M_0(z) e^{-i\Delta\omega T/2} e^{i\gamma G z T/2}$$

Next, we acquire in the presence of a gradient. During that time (starting at t=0),

$$M_{xy}(z,t) = M_0(z) e^{-i\Delta\omega(T/2+t)} e^{i\gamma G z(T/2-t)}$$

and hence, the acquired signal, as a function of time, will be:

$$s(t) \propto \int_{\text{sample}} M_{xy}(z,t)\,dz = e^{-i\Delta\omega(t+T/2)} \int_{\text{sample}} M_0(z) e^{-i\gamma G z(t-T/2)}\,dz$$

We change variables by substituting $k(t) = \gamma G(t - T/2)$, so

$$s(k) = s(t(k)) \propto e^{-i\left(\frac{\Delta\omega k}{\gamma G} + \Delta\omega T\right)} \int_{\text{sample}} M_0(z) e^{-izk(t)}\,dz, \quad -\frac{\gamma G T}{2} \le k \le +\frac{\gamma G T}{2}$$

This can be written as:

$$s(\Delta\omega,k)=e^{-i\frac{\Delta\omega k}{\gamma G}}e^{-i\Delta\omega T}s(\Delta\omega=0,k)$$

So it's the same signal as we would've gotten from the Δω=0 case, with (i.) a constant phase (the $e^{-i\Delta\omega T}$) and a linear phase ($e^{=i\Delta\omega k/\gamma G}$) multiplying it. The original image is recovered by inverse-Fourier-transforming over the k coordinate:

$$\begin{aligned} M_0^{(\text{reconstructed})}(\Delta\omega,z) &= \int s(\Delta\omega,k)e^{ikz}dk \\ &= e^{-i\Delta\omega T}\int e^{-i\Delta\omega k/\gamma G}s(\Delta\omega=0,k)e^{ikz}dk \\ &= e^{-i\Delta\omega T}\int s(\Delta\omega=0,k)e^{ik\left[z-\frac{\Delta\omega}{\gamma G}\right]}dk \\ &= e^{-i\Delta\omega T}M_0\left(z-\frac{\Delta\omega}{\gamma G}\right) \end{aligned}$$

since $M_0(z)=\int s(\Delta\omega=0,k)e^{ikz}dk$. So you get (i.) a position shift, and (ii.) a phase shift as well (a complex number of magnitude 1, $e^{-i\Delta\omega T}$, multiplying every voxel in the image).

**Question 3**

| Increase ... | FOV | N | Δz | Notes |
|---|---|---|---|---|
| G | - | 0 | - | Δk, $k_{max}$ increase (while $k_{max}/\Delta k=N=const$). Hence FOV & Δz decrease. |
| T (keeping N fixed) | - | 0 | - | $k_{max}$ increases, Δk increases. |
| N (keeping T fixed) | + | + | 0 | $k_{max}$ stays the same, Δk decreases. |
| Δt (keeping N fixed) | - | 0 | - | N is fixed, so an increase in Δt implies an increase in T as well. Thus, $k_{max}$ and Δk increase. |

+ means it increases, 0 means no change, - means a decrease.