Question 1.

Thinking in terms of a trajectory in k-space (see figure on right), we must have $\Delta k_x = \Delta k_y = 1/\text{FOV}_x = 1/\text{FOV}_y = 1/L$. Now,

$$
\Delta k_x = \gamma G_x \times \text{(dwell time)} = \gamma G_x \times \frac{T_x}{N}
$$

$$
\Delta k_y = \gamma G_y \Delta t_y
$$

where the dwell time – the time between each point sampled – is given by $T_x$ (the time it takes to sample a line) divided by $N$ (the number of points in a line). Since, by choice, $\Delta k_x = \Delta k_y$, one obtains

$$
\Delta k_x = \gamma G_x \frac{T_x}{N} = \gamma G_y \Delta t_y = \Delta k_y
$$

and since, by assumption, $T_x/N = \Delta t_y$, the desired result ($G_x = G_y$) is obtained.

Question 2.

Think of what it means to acquire a 1D image of an object containing water, having some offset, $\Delta \omega$. Let’s take the water profile in the question for concreteness:

In the absence of a gradient, all water spins have the same frequency, $\Delta \omega$. Acquiring (without a gradient) and Fourier-transforming would yield a peak at $\Delta \omega$: 
When you switch on a gradient, spins get a frequency assigned to them as a function of position: \( \omega = \gamma Gz + \Delta \omega \). Acquiring in the presence of this gradient means you will get a peak from each position in the sample, which will give you the 1D profile of your object; the peak at \( \omega \) will come from the spins at \( z = \frac{\omega - \Delta \omega}{\gamma G} \).

How would you translate this frequency axis to a position axis? By inverting the relation \( \omega = \gamma Gz + \Delta \omega \) to yield \( z(\omega) = \frac{\omega - \Delta \omega}{\gamma G} \). If you know \( \Delta \omega \) (you do) and \( G \), you can recover \( z \).

For a sample with both fat and water, with \( \Delta \omega = 0 \) for the water and \( \Delta \omega \neq 0 \) for the fat (by assumption), one would get the following result (in frequency space) after Fourier transforming the signal:

You can see there’s a \( \Delta \omega \) gap in frequency space between the two profiles. When converting frequency to position, however, you have a problem. You don’t know a-priori which peak comes from fat and which from water (here I’ve drawn the object so we do know a-priori which peak comes from fat and which peak comes from water, but when imaging unknown objects it’s impossible to say). Therefore, we don’t know \( \Delta \omega \). We must assume \( \Delta \omega = 0 \) (out of ignorance), so

\[
\omega = \gamma Gz \quad \Leftrightarrow \quad z = \frac{\omega}{\gamma G}
\]
Using this to "translate" between frequency & position, we obtain

\[
\frac{\Delta \omega}{\gamma G} - L \quad \quad \frac{\Delta \omega}{\gamma G} \quad \quad 0 \quad \quad 0 \quad \quad L
\]

So the shift, when expressed in terms of position, becomes

\[
\Delta z = \frac{\Delta \omega}{\gamma G}
\]

which is the magnitude of the shift between the fat and the water. The obvious way to reduce this artifact is to increase the gradient, G (making \(\Delta z\) smaller).

There is also a formal, mathematical solution to the problem. Let \(M_0(z)\) be the distribution of water or fat in the imaged object, with some chemical shift \(\Delta \omega\). After exciting the spins, we have

\[
M_{xy}(z) = M_0(z)
\]

We apply a gradient \(-G\) for a time \(T/2\), during which \(\omega(z) = \Delta \omega - \gamma G z\), and after which

\[
M_{xy}(z) = M_0(z)e^{-i\omega(z)T/2} = M_0(z)e^{-i\Delta \omega T/2}e^{-i\gamma G z T/2}
\]

Next, we acquire in the presence of a gradient. During that time (starting at \(t=0\)),

\[
M_{xy}(z,t) = M_0(z)e^{-i\Delta \omega (T/2+t)}e^{-i\gamma G z (T/2-t)}
\]

and hence, the acquired signal, as a function of time, will be:

\[
s(t) x \int_{\text{sample}} M_{xy}(z,t)dz = e^{-i\Delta \omega (t+T/2)} \int_{\text{sample}} M_0(z)e^{-i\gamma G (t-T/2)}dz
\]

We change variables by substituting \(k(t) = \gamma G (t-T/2)\), so

\[
s(k) = s(t(k)) x e^{-i\Delta \omega t/\gamma G} \int_{\text{sample}} M_0(z)e^{-i\gamma G k(t)}dz, \quad -\frac{\gamma GV}{2} \leq k(\tau) \leq \frac{\gamma GV}{2}
\]
This can be written as:

\[ s(\Delta \omega, k) = e^{-i\frac{\Delta \omega k}{\gamma G}} e^{-i\Delta \omega T} s(\Delta \omega = 0, k) \]

So it’s the same signal as we would’ve gotten from the \( \Delta \omega = 0 \) case, with (i.) a constant phase (the \( e^{i\Delta \omega T} \)) and a linear phase (\( e^{-i\frac{\Delta \omega k}{\gamma G}} \)) multiplying it. The original image is recovered by inverse-Fourier-transforming over the \( k \) coordinate:

\[
M_{0, \text{reconstructed}}(\Delta \omega, z) = \int s(\Delta \omega, k) e^{ikz} dk \\
= e^{-i\Delta \omega T} \int e^{-i\Delta \omega k / \gamma G} s(\Delta \omega = 0, k) e^{ikz} dk \\
= e^{-i\Delta \omega T} \int s(\Delta \omega = 0, k) e^{ikz} \left[ \frac{\Delta \omega}{\gamma G} \right] dk \\
= e^{-i\Delta \omega T} M_0 \left( z - \frac{\Delta \omega}{\gamma G} \right)
\]

since \( M_0(z) = \int s(\Delta \omega = 0, k) e^{ikz} dk \). So you get (i.) a position shift, and (ii.) a phase shift as well (a complex number of magnitude 1, \( e^{i\Delta \omega T} \), multiplying every voxel in the image).

**Question 3**

<table>
<thead>
<tr>
<th>Increase ...</th>
<th>FOV</th>
<th>N</th>
<th>( \Delta z )</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G )</td>
<td>-</td>
<td>0</td>
<td>-</td>
<td>( \Delta k, k_{\text{max}} ) increase (while ( k_{\text{max}} / \Delta k = N = \text{const} )). Hence FOV &amp; ( \Delta z ) decrease.</td>
</tr>
<tr>
<td>( T ) (keeping N fixed)</td>
<td>-</td>
<td>0</td>
<td>-</td>
<td>( k_{\text{max}} ) increases, ( \Delta k ) increases.</td>
</tr>
<tr>
<td>( N ) (keeping T fixed)</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>( k_{\text{max}} ) stays the same, ( \Delta k ) decreases.</td>
</tr>
<tr>
<td>( \Delta t ) (keeping N fixed)</td>
<td>-</td>
<td>0</td>
<td>-</td>
<td>N is fixed, so an increase in ( \Delta t ) implies an increase in ( T ) as well. Thus, ( k_{\text{max}} ) and ( \Delta k ) increase.</td>
</tr>
</tbody>
</table>

+ means it increases, 0 means no change, - means a decrease.