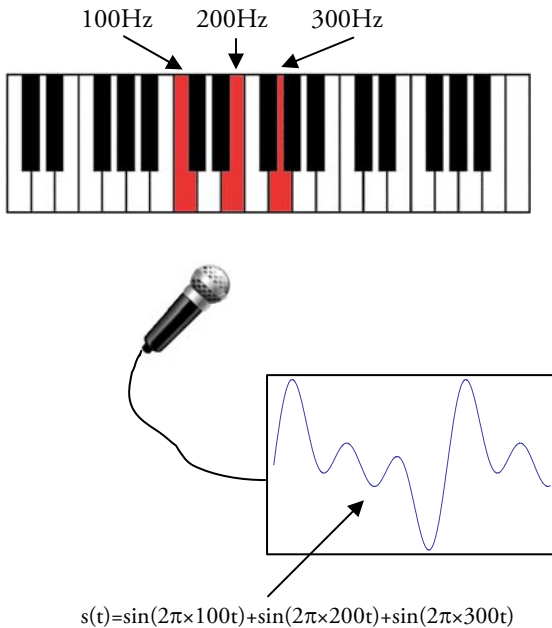# V
The Fourier Transform

Lecture notes by Assaf Tal

1. Motivation

Imagine playing three notes on the piano, recording them (storing them as a .wav or .mp3 file), and then plotting the resulting waveform on the computer:

s(t)=sin(2π×100t)+sin(2π×200t)+sin(2π×300t)

Note: those aren't the actual frequencies of those notes on a piano, I've just chosen simple numbers for the example.

Looking at the resulting waveform really doesn't tell you anything about the frequencies of the notes that were played. Can we recover them? The answer is yes, using the so-called *Fourier transform*. The Fourier transform (FT) is a "black box" that tells you exactly what periodicities are present in your signal.

2. Definition

Given a function f(t), its Fourier transform is a function $\hat{f}(\omega)$, defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \qquad \text{(FT)}$$

It can be shown that, given $\hat{f}(\omega)$, the function f(t) can be recovered using the *inverse Fourier transform*:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}dt \qquad \text{(Inverse FT)}$$

Some notes:

1. Other books may define the FT slightly differently: sometimes the FT might have a 1/2π factor in front of it, or the exponent might have a plus sign. The only two things that will never vary are: (i.) the product of the factors in front of the integral of the FT and the inverse FT will always equal 1/2π. (ii.) If the FT has a minus sign in the exponent, the IFT will have a plus sign, and vice versa.

2. Don't be afraid of the $e^{-i\omega t}$ factor. You can treat it as you would any other exponent (now, if you're afraid of exponents, that's another problem). Examples:

   a. $\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t}$

   b. $\int e^{i\omega t}dt = \frac{1}{i\omega}e^{i\omega t}$

   c. $e^{i\omega t}e^{i\omega' t} = e^{i\omega t + i\omega' t} = e^{i(\omega+\omega')t}$

3. As you've noticed, $\hat{f}(\omega)$ might be a complex valued function, e.g. $\hat{f}(3) = 5 - 2i$. Therefore it has both a magnitude and a phase as any other complex number: $\hat{f}(\omega) = |\hat{f}(\omega)|e^{i\phi(\omega)}$.

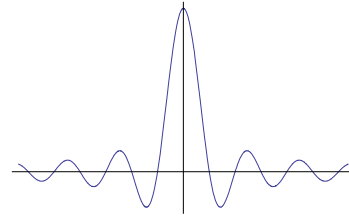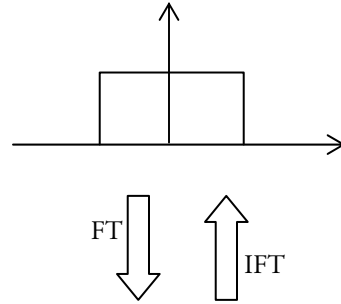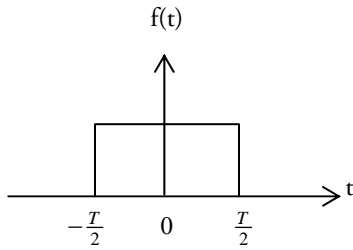You can think of the FT as a black box, that takes one function f(t) and spits out another, $\hat{f}(\omega)$:
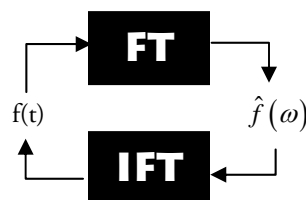
3. Computation

3.1 Example #1

Ok, enough talk. Let's work out a simple example. Let's take a rectangular function:

$$f(t) = \begin{cases} 1 & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

f(t)

We compute:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= \int_{-T/2}^{T/2} e^{-i\omega t} dt$$

$$= \frac{e^{-i\omega t}}{-i\omega} \bigg]_{t=-T/2}^{t=T/2}$$

$$= \frac{e^{-i\omega T/2} - e^{i\omega T/2}}{-i\omega}$$

$$= \frac{T}{2} \frac{e^{i\omega T/2} - e^{-i\omega T/2}}{i\omega \times \frac{T}{2}}$$

Now I'm going to use an identity:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

with $x = \omega T / 2$ and obtain:

$$\hat{f}(\omega) = T \frac{\sin\left(\frac{\omega T}{2}\right)}{\omega T / 2} \equiv T \text{sinc}\left(\frac{\omega T}{2}\right)$$

where sinc(x) = sin(x)/x by definition. We plot the results:

If we'll take the resulting sinc function and apply the IFT to it we'll get the rectangular function back. However, that calculation is a bit tricky and requires complex calculus, so we won't attempt it here; you'll just have to take my word for it that it works.

3.2 Example #2

Going back to the motivating example in section (2.1), remember we got a signal s(t) that didn't really tell us much about the frequencies that made it up:
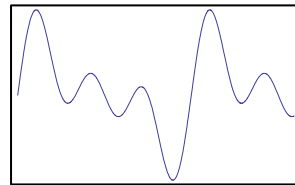
If we were to apply the FT to it, we would obtain something that'd look like this:
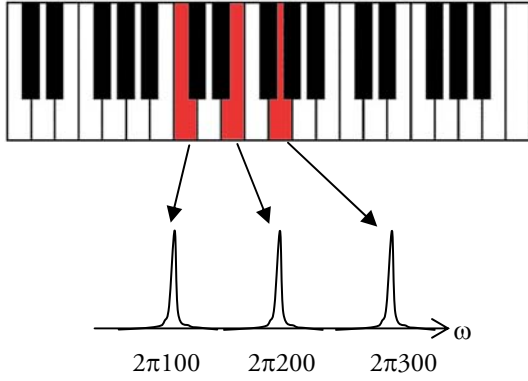
The ω axis is 2π times the frequencies of the original notes (ω is sometimes called the angular frequency). Each "peak" would tell us that a particular note on the piano were present. The peaks are independent in the sense that, if we were to drop the middle note, the middle peak would disappear but the other two would be unaffected. The height of the peaks indicates how strong we hit each key (the loudness of each frequency) – in this case I've assumed all keys were struck with even force.

4. The 2D/3D Fourier Transform

The FT we've encountered so far was one dimensional. That is, it took it a function of one variable and gave us back a function of one variable:

$$f(t) \rightarrow \text{FT} \rightarrow \hat{f}(\omega)$$

There is also a 2D FT, that takes as its input a function of two variables and returns another function of two variables; and a 3D FT, that takes as its input a function of three variables, x, y & z, and returns another function of three variables, k_x, k_y, k_z:

$$f(\mathbf{r}) \rightarrow \text{FT} \rightarrow f(\mathbf{k})$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{k} = (k_x, k_y, k_z)$ are coordinates on a 3D grid. The 3D FT is defined as:

$$\hat{f}(\mathbf{k}) = \int f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}$$

$$\equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) e^{-ik_x x} e^{-ik_y y} e^{-ik_z z} dxdydz$$

Here $\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z$ is the scalar product of the two vectors, $\mathbf{k}$ and $\mathbf{r}$. Contrast this with the 1D FT:

$$\hat{f}(\omega) = \int f(t) e^{-i\omega t} dt$$

The 3D FT behaves in exactly the same manner as the 1D FT: it "detects" periodicities in the function $f(\mathbf{r})$. However, just looking at the 2D (or 3D) FT of a 2D (or 3D) function rarely tells you anything about it. For example, here are two totally distinct images, and the logs of the magnitude of their corresponding FTs (the magnitude of the FT has a wide dynamic range, so one often looks at its log):
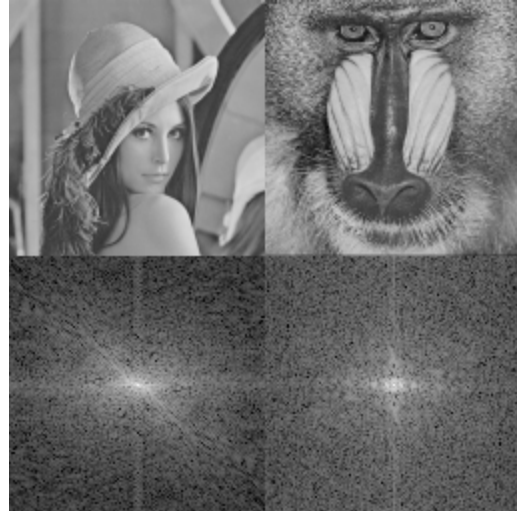

An image, by the way, can be regarded as a 2D function, f(x,y), with f(x₀,y₀) being the brightness of the pixel at (x_0, y_0).

The inverse FT, in 3D, is given by:

$$f(\mathbf{r}) = (2\pi)^{-3} \int \hat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}$$

$$\equiv (2\pi)^{-3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(k_x, k_y, k_z) e^{ik_x x} e^{ik_y y} e^{ik_z z} dxdydz$$

Let's think for a moment in 2D. We can think of the 2D FT, f(**k**)=f(k_x,k_y), as a function defined in the k_x-k_y plane. The values of f(**k**) at the center of this "k-plane" represent the "slowly-varying frequencies" of the function, and are hence responsible for the gross-features of the image; conversely, the values of f(**k**) away from the center represent the "fast components" of the image, and hence usually they make up the edges and sharp features of the image. For example, here is a picture with just the center of the k-plane retained:
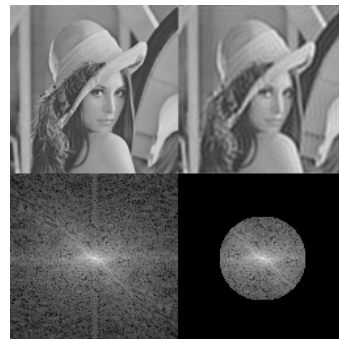
Note how blurred the image seems: the sharp features (the edges) have become less defined, because we've removed the "fast" components of the image. Likewise, by removing mainly the center of the k-plane:
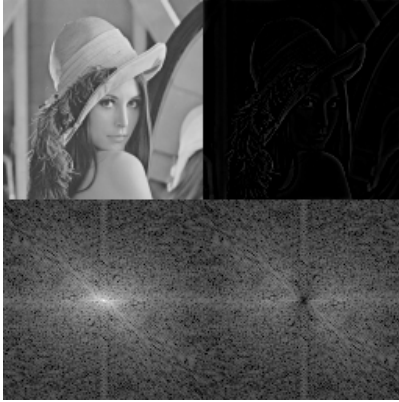
we remain with just the "fast", discontinuous components, such as the edges.