IV
Field Gradients

Lecture notes by Assaf Tal

So far we've seen how spins behave:

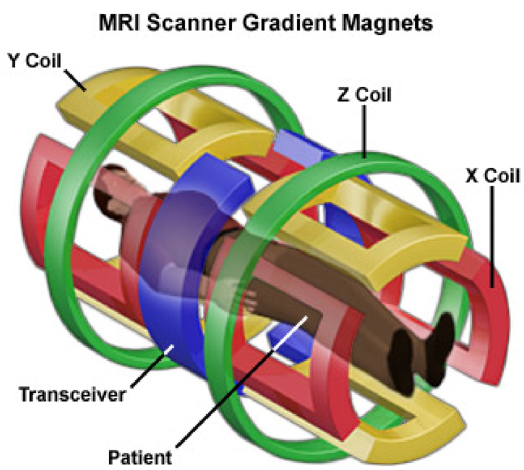
1. When the RF is off, they precess in the xy-plane according to their offset, and eventually relax back to equilibrium via T_1 and T_2^* decay.
2. When the RF is turned on, they precess about the effective field (RF + offset). We can give "hard" π, $\pi/2$ and any-other-tilt-angle pulses about any axis in the xy-plane.

In this lecture I'll introduce an additional tool for manipulating the spins: the gradient coils. These will tie together space and frequency and allow us to image objects.

1. The Gradient Coils

1.1 Qualitative Description

The main MRI bore has three coils around it, capable of generating an linearly increasing z-field along the x, y and z axes:



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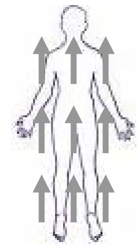
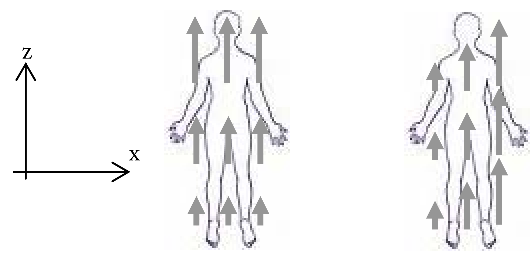
The linear field gradients are creating by pumping current through these coils. Ampere's law tells us that flowing current creates magnetic fields around it.

It is important to understand visually what is meant by the gradient coils. Here is a 2D illustration:



z-field (a.k.a. offset) w/o gradients



w/ z-gradient w/ x-gradient

Note: in the absence of RF irradiation, the field always points in the z-direction!

1.2 Quantitative Description

Spin physics is quite straightforward: generate a magnetic field, and the spins will precess about it. Up until now, the fields involved in our discussion were, in the rotating frame, (i) the offset, and (ii) the RF:

$$\mathbf{B}(t) = \underbrace{\tfrac{\Delta\omega}{\gamma}\hat{\mathbf{z}}}_{\text{Offset}} + \underbrace{\mathbf{B}_{RF}(t)}_{\text{RF}}$$

The gradient field allows us to add a third term:

$$\mathbf{B}(t) = \Big(\tfrac{\Delta\omega}{\gamma} + \underbrace{\mathbf{G}(t)\cdot\mathbf{r}}_{\text{Gradient field}}\Big)\hat{\mathbf{z}} + \mathbf{B}_{RF}(t)$$

The quantity **G**(t) is called the gradient field and is completely controlled by us, the scientists, via the hardware's console. Explicitly,

$$\left(\mathbf{G}(t)\cdot\mathbf{r}\right)\hat{\mathbf{z}} = \left[\underbrace{G_x}_{\text{x-gradient}}(t)x + \underbrace{G_y}_{\text{y-gradient}}(t)y + \underbrace{G_z}_{\text{z-gradient}}(t)z\right]\hat{\mathbf{z}}$$

Notes:

1. We can control each term, $G_x(t)$, $G_y(t)$, $G_z(t)$ and *shape* it as we wish.
2. Note that, e.g., the x-gradient does **not** create a field along the x-axis. Rather, it increases/decreases the z-field along the x-axis. See the pictures above for a clarification.
3. The gradients G_k are measured in field/unit length. Usually they're specified in mT/m or G/cm. The 3T Siemens Trio we have goes up to 45 mT/m. Non-human MRIs go up much higher, since they're not subject to the same safety considerations as those of human magnets (animals, alas, can't sue).
4. **r** is the position of the spin. Different spins will have different positions (different **r** values), and hence will experience a different z-component of the field: $B_z\left(\mathbf{r}\right) = \frac{\Delta\omega}{\gamma} + \mathbf{G}(t)\cdot\mathbf{r}$.

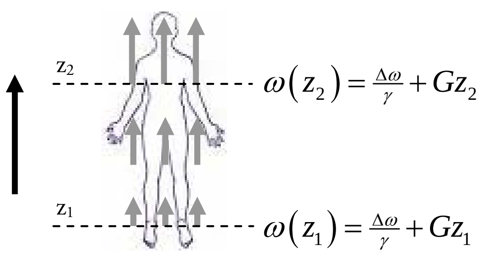
2. The Effect of a Gradient

The gradient (when constant), in effect, assigns a linearly increasing offset (i.e. field in the z-direction) to the spins in the sample. Consider, for example, a constant z-gradient, the field in the rotating frame would be:

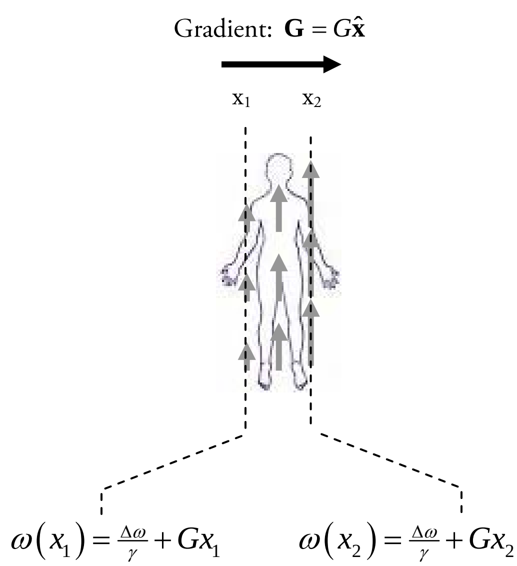
$$\mathbf{B}(t) = \underbrace{\left(\tfrac{\Delta\omega}{\gamma} + Gz\right)}_{B_z}\hat{\mathbf{z}} + \mathbf{B}_{RF}(t)$$

and the angular frequency:

$$\omega(z) = \gamma B_Z = \Delta\omega + \gamma G z$$



As another example, consider the x-gradient (assuming left-right on this page corresponds to the x-axis):



2.2 Quantitative Description: Constant Gradients

We turn next to a quantitative description of the effect of the gradients on a bunch of spins. First, at equilibrium, the gradients have no effect: since the spins are parallel to the main field, and since the gradient coils merely change the value of this main field (and not its direction), nothing happens to the spins[1].

It is only once we tip the spins onto the xy-plane using a $\pi/2$ pulse (or some other tilt angle) that the gradient has any effect. Let's suppose we have an object with a uniform magnetization density, M_0. Thus, at equilibrium,

$$\mathbf{M} = M_0\hat{\mathbf{z}}\,.$$

Once we've tipped this magnetization using a $\pi/2$ pulse (say, 90_{-y}), we end up with this magnetization along the x-axis:

$$\mathbf{M} = M_0\hat{\mathbf{x}}\,.$$

At this point, if left alone, and if we neglect relaxation, the spin will precess about the local field (which is along the z-axis); that is, about the offset $\Delta\omega$:

$$\mathbf{M}(t) = \begin{pmatrix} M_0\cos(\Delta\omega t) \\ -M_0\sin(\Delta\omega t) \\ 0 \end{pmatrix}.$$

If we now turn on a gradient $\mathbf{G}$ in addition to the offset (whatever it may be), spins at different points, $\mathbf{r}=(x,y,z)$, will now precess with different angular velocities:

$$\mathbf{M}(\mathbf{r},t) = \begin{pmatrix} M_0\cos[(\Delta\omega + \gamma\mathbf{G}\cdot\mathbf{r})t] \\ -M_0\sin[(\Delta\omega + \gamma\mathbf{G}\cdot\mathbf{r})t] \\ 0 \end{pmatrix}.$$

Note:

[1] There is a completely negligible thermodynamic effect here: the equilibrium population is proportional to the main magnetic field, B_0, which is not a constant now. However, this effect is extremely minor and will be disregarded.

1. **M** is now a function of both position (**r**) and time (t).
2. Note the minus sign of the y-component. This is because the spin precesses according to the **left hand rule**.
3. **Terminology:** the term $\Delta\omega$ is there because of the chemical shift of the spin. The term $\gamma \mathbf{G}\cdot\mathbf{r}$ is the result of the gradient. The entire sum, $\Delta\omega + \gamma\mathbf{G}\cdot\mathbf{r}$, is referred to (as a whole) the spin's **offset**. It is position dependent. In general, the **offset** is the total z-field felt by the spin in the rotating frame (chemical shift + gradient), specified in units of frequency. In other words, the z-field felt by the spin at **r** is

$$\text{field} = \frac{\text{offset}}{\gamma} = \frac{\Delta\omega + \gamma\mathbf{G}\cdot\mathbf{r}}{\gamma}$$

It's convenient to switch at this point to complex notation:

$$\begin{aligned} M_{xy} &= M_x + iM_y \\ &= M_0 \cos\left[\left(\Delta\omega + \gamma\mathbf{G}\cdot\mathbf{r}\right)t\right] \\ &\quad - iM_0 \sin\left[\left(\Delta\omega + \gamma\mathbf{G}\cdot\mathbf{r}\right)t\right] \\ &= M_0 e^{-i\left(\Delta\omega+\gamma\mathbf{G}\cdot\mathbf{r}\right)t} \end{aligned}$$

Furthermore, M_0 tends to vary from point to point in the imaged object, so $M_0=M_0(\mathbf{r})$. Thus:

$$M_{xy} = M_0\left(\mathbf{r}\right)e^{-i\left(\Delta\omega+\gamma\mathbf{G}\cdot\mathbf{r}\right)t}.$$

To add the effect of relaxation on the transverse magnetization, we need merely to add a decaying term to both M_x and M_y (with a time constant T_2, since the magnetization is transverse):

$$\begin{aligned} M_x &\to M_x e^{-t/T_2} \\ M_y &\to M_y e^{-t/T_2} \end{aligned}$$

This means that

$$\begin{aligned} M_{xy} &= M_x + iM_y \\ &\to M_x e^{-t/T_2} + iM_y e^{-t/T_2} \\ &= \left(M_x + iM_y\right)e^{-t/T_2} \end{aligned}$$

That is, we merely need to add a factor $e^{-t/T2}$ to our previous result:

$$\boxed{M_{xy}\left(\mathbf{r},t\right) = M_0\left(\mathbf{r}\right)e^{-i\left(\Delta\omega+\gamma\mathbf{G}\cdot\mathbf{r}\right)t - t/T_2}}$$

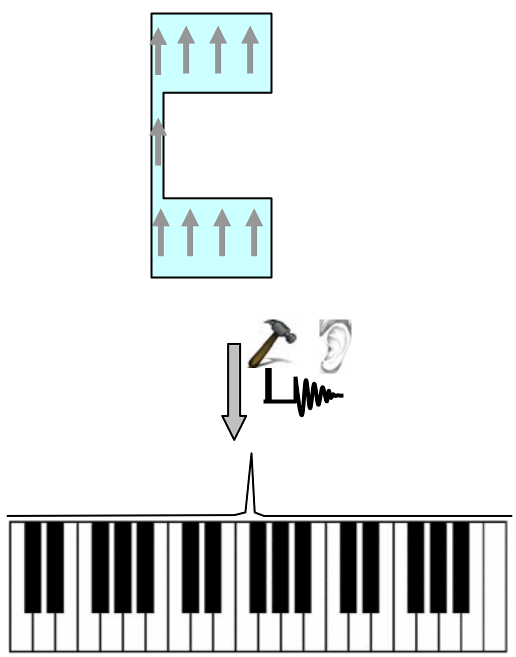
True for a constant gradient
(not a time-dependent one!)

3. A Basic Imaging Sequence

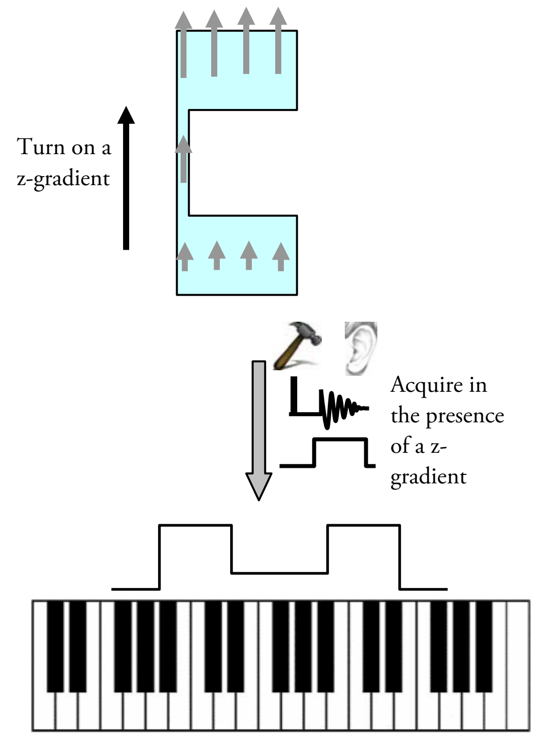
Imaging requires some formal machinery, and I'd like to motivate it instead of just dropping it on you. To this end, recall the basic spectroscopy experiment we've described in lecture 2:

The signal measured is $\Lambda\sin(\Delta\omega t)e^{-t/T2}$. The periodicity of the signal tells us the offset of the spins, while its intensity is proportional to the number of spins (double the number, and the intensity will double as well).

Remember the analogy we drew in chapter 2 to the piano: the π/2 pulse was likened to a hammer striking a black box, and the acquisition was to listening to the audible resonances. In the case of spectroscopy of a water sample, all molecules had the same precession frequency, so we heard just one frequency (I've taken an odd shaped sample for reasons that will become clear shortly):



If we were to turn on our z-gradient, we'd be creating a linearly increasing offset. This means the higher up (*z*-wise) the spins are positioned, the higher the "pitch" they'll emit. Furthermore, loudness of each pitch will be determined by the number of



Thus, we "hear" the shape of the sample! That's the idea behind imaging: turn on a gradient, assign a different offset to different positions, excite the sample and "listen". The presence, and intensity, of different frequencies will tell you how many spins (which are proportional to the number of water molecules) you have at each position.

Of course, reality is more complicated:

1. What does it mean to "hear" spins?
2. Gradients increase the offset along one direction, so they're one dimensional. How can we use them to image 2D and 3D objects?
3. The acquired signal s(t) will originate from all spins simultaneously. Different spins will have different frequencies. How can we tell apart the different frequencies in our signal? How can we tell how many protons there are of each frequency?

The answer to #3 is the Fourier Transform, which I'll introduce in the next lecture.