MRI Primer, Exercise #3
Due 15/Dec/2009

1. **"Hard" Pulses**. An RF pulse is said to be "hard" if one can neglect the offset when computing its effects; in other words, when the offset is negligible compared to the applied RF field. Thus, **during a hard pulse, all spins are assumed to be on-resonance**. Suppose you have a bunch of proton spins with different chemical shifts, having different offsets in the rotating frame, ranging from +100 Hertz to -100 Hertz. You wish, to a good approximation, to excite all those spins from thermal equilibrium along the z-axis onto the x-axis using a 90_{-y} RF pulse. Give a condition on the pulse's **duration** for it to be considered hard (i.e. the duration must be must longer than ... or much shorter than ... something). Hint: the answer is short.

2. **Precession + Relaxation**. The Bloch equations are:

$$\begin{cases} \frac{dM_x}{dt} = \gamma M_y B_z - \gamma M_z B_y - \frac{M_x}{T_2} \\ \frac{dM_y}{dt} = \gamma M_z B_x - \gamma M_x B_z - \frac{M_y}{T_2} \\ \frac{dM_z}{dt} = \gamma M_x B_y - \gamma M_y B_x - \frac{M_z - M_0}{T_1} \end{cases}$$

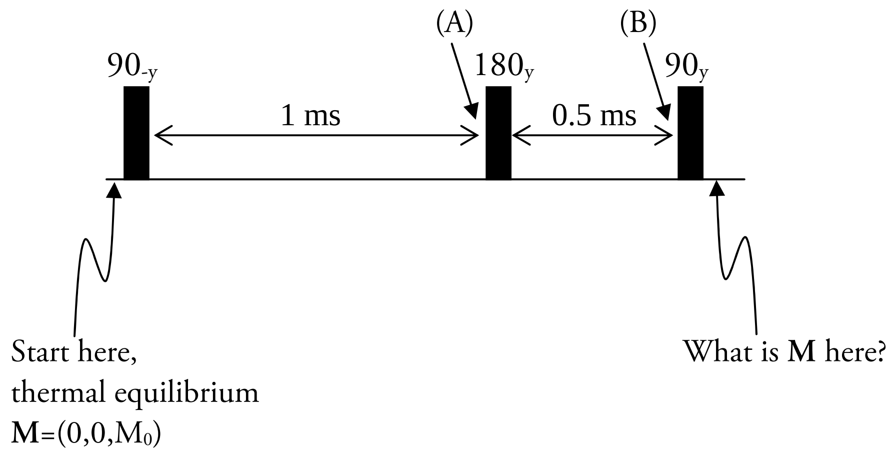
   a. Show that, if the initial magnetization is $\mathbf{M}_i=(M_a,0,M_b)$, no RF is present, and the offset is ΔB, the solution to the equations (and hence the magnetization, as a function of time) is given by:

$$\mathbf{M}(t) = \begin{pmatrix} M_a \cos(\Delta\omega t) e^{-t/T_2} \\ -M_a \sin(\Delta\omega t) e^{-t/T_2} \\ M_0\left[1 - e^{-t/T_1}\right] + M_b e^{-t/T_1} \end{pmatrix}$$

   where $\Delta\omega=\gamma\Delta B$. Hint: there really isn't too much to think about. Just (i.) make sure $\mathbf{M}(t=0) = \mathbf{M}_i$ and (ii.) differentiate the components of $\mathbf{M}(t)$ and make sure each satisfies the appropriate differential equation (there's some dirty work involved).

   b. Show that, using complex notation, $M_{xy}(t)=M_a e^{-i\Delta\omega t-t/T2}$ (recall $M_{xy}(t)=M_x(t)+iM_y(t)$). Once again, this is just a bit of dirty work (use Euler's identity, $e^{ix} = \cos(x) + i\sin(x)$). We'll be using this result in class.

3. **Analyzing a Pulse Sequence**. A pulse sequence is just a succession of RF pulses with delays between them. Suppose you're given a spin with an offset of 1 kHz (this is without the 2π!), and suppose the RF pulses shown below are **hard**, meaning they are strong enough for you to neglect the spin's offset during each RF pulse (but not in-between pulses!). What will the magnetization vector be at the end of the sequence, assuming you start out from thermal equilibrium, $\mathbf{M}=(0,0,M_0)$? Justify your answer by writing down **M** in the 2 intermediate steps along the way marked by (A), just before the 180_y, and (B), just before 90_y.



Neglect relaxation effects (why is this is an acceptable approximation? Hint: take a look at the table of T_2 and T_1 values in question #5, exercise #2).
Remember: "hard" pulses are short, strong RF pulses, which render the offset negligible throughout their duration.

4. **The Fourier Transform**. In this question we're going to practice computing the Fourier Transform (FT).
   a. Derive the identities:

$$\sin(x)=\frac{e^{ix}-e^{-ix}}{2i}, \qquad \cos(x)=\frac{e^{ix}+e^{-ix}}{2}$$

   Hint: use Euler's identity, $e^{ix}=\cos(x)+i\sin(x)$.

   b. Compute the 2D FT, $\hat{f}(k_x,k_y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(k_x x+k_y y)}dxdy$, of the following function, and show that it's real (use the results of part (a)):

$$f(x,y)=\begin{cases}1 & -a\le x\le a \text{ and } -a\le y\le a\\ 0 & \text{otherwise}\end{cases}$$